

# Generalized Vaidya Solutions and Misner-Sharp mass for $n$ -dimensional massive gravity

Ya-Peng Hu<sup>1,2,3</sup> \*, Xin-Meng Wu<sup>1</sup> †, Hongsheng Zhang<sup>3,4</sup> ‡

<sup>1</sup> *College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China*

<sup>2</sup> *Instituut-Lorentz for Theoretical Physics, Leiden University,  
Niels Bohrweg 2, Leiden 2333 CA, The Netherlands*

<sup>3</sup> *Key Laboratory of Theoretical Physics, Institute of Theoretical Physics,  
Chinese Academy of Sciences, Beijing, 100190, China*

<sup>4</sup> *Center for Astrophysics, Shanghai Normal University, 100 Guilin Road, Shanghai 200234, China*

Dynamical solutions are always of interests in gravity theories. We derive a series of generalized Vaidya solutions in the  $n$ -dimensional de Rham-Gabadadze-Tolley (dRGT) massive gravity with a singular reference metric. Similar to the case of the Einstein gravity, the generalized Vaidya solution can describe shining/absorbing stars. Moreover, we also find a more general Vaidya-like solution by introducing a more generic matter field than the pure radiation in the original Vaidya solution. As a result, the above generalized Vaidya solution is naturally included in this Vaidya-like solution as a special case. We investigate the thermodynamics for this Vaidya-like spacetime by using the unified first law, and present the generalized Misner-Sharp mass. Our results show that the generalized Misner-Sharp mass does exist in this spacetime. In addition, the usual Clausius relation  $\delta Q = TdS$  holds on the apparent horizon, which implicates that the massive gravity is in a thermodynamic equilibrium state. We find that the work density vanishes for the generalized Vaidya solution, while it appears in the more general Vaidya-like solution. Furthermore, the covariant generalized Misner-Sharp mass in the  $n$ -dimensional de Rham-Gabadadze-Tolley massive gravity has also been further derived by taking a general metric ansatz into account.

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## I. INTRODUCTION

Massive gravity is a significant and fundamental extension of the Einstein gravity. But, in opposite to our intuition, to endow a mass to the graviton is not an easy problem. In 1939, Fierz and Pauli first introduced the linear massive gravity theory [1]. Note that, a massless graviton has only two polarizations, and a sound massive gravity theory generally has five degree of freedoms. However, the surplus three degree of freedoms are proved to be intractable when the mass of graviton vanishes in the linear massive gravity [2]. To overcome this problem, one tries to introduce the non-linear massive gravities, but a more serious problem, the Boulware-Deser ghost problem, appears[3]. Recently, the so called de Rham-Gabadadze-Tolley (dRGT) massive gravity has been proposed [4–6], which is a nonlinear massive gravity theory and has been found to be ghost-free [7, 8]. Note that, in the dRGT model, the reference metric is usually full rank. But yet, a singular reference metric is also found to be important [9], in which the ghost problem is investigated in [10, 11]. Because in the AdS/CFT scenario [12–14], many clues show that the massive graviton in the bulk is related to some interesting effects of the dual field which resides on the UV boundary of an asymptotical AdS spacetime, i.e. effects like a lattice to deduce the momentum dissipation [9, 15–17]. Many researches about the dRGT massive gravity have been done until now [9–11, 15–28].

Among these researches, one interesting issue is to find out the exact solutions in the dRGT massive gravity[18–21], since exact solutions take pivotal positions in physics, especially in gravity theory with the high non-linearities of the field equations. Usually we assume some symmetries of the spacetime when we seek a new solution. The translation invariance along a time-like Killing vector is one of the most important symmetry. Except cosmology and problems about gravitational waves, almost all studies in astrophysics are based on the time symmetry, for examples in the cases of compact stars, accretion disks, even supernovae, where usually a Schwarzschild or Kerr geometry is assumed. But in some violent astrophysical processes, or the mass of the matters surrounding the central celestial bodies are not negligible, such an assumption may be no longer reliable. However, a dynamical solution describing such realistic processes is proved to be a very difficult topic.

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\* Electronic address: huyp@nuaa.edu.cn

† Electronic address: wuxm@nuaa.edu.cn

‡ Electronic address: hongsheng@shnu.edu.cn

Vaidya found an important dynamical toy model for a spherically symmetric spacetime [29],

$$ds^2 = -\left(1 - \frac{2M(v)}{r}\right)dv^2 + 2dvdr + r^2 d\Omega_2^2, \quad (1)$$

where  $M(v)$  is the mass parameter,  $d\Omega_2^2$  is the metric on the 2-dimensional unit sphere, and the stress tensor is given by  $T_{ab} = \mu l_a l_b$ , here  $l_a = (dv)_a$  in the above coordinates  $(v, r, x^i)$  and  $\mu$  is the energy density. Now this solution is well-known as the Vaidya solution. Note that, the Vaidya solution describes a spherically symmetric spacetime sourced by massless particles (not quanta of the Maxwell fields), which are called pure radiations. In addition, since  $M(v)$  is an undetermined function in the Vaidya metric, in principle, it can describe an arbitrary spherically symmetric energy flow from the central star. When  $M(v)=\text{constant}$  it comes back to the Schwarzschild one, and when  $M(v) = 0$  it degenerates to the Minkowski. It should be emphasized that the Vaidya solution is an important solution since it encodes some essential properties of the dynamical spherically symmetric, while keeps simple enough to handle. Therefore, in our paper the first task is to generalize the above Vaidya solution to a more general case, i.e. exact generalized Vaidya solutions in the  $n$ -dimensional spacetime with maximally symmetric inner space in the dRGT massive gravity, and the metric ansatz is

$$ds^2 = -f(v, r)dv^2 + 2dvdr + r^2 \gamma_{ij} dx^i dx^j, \quad (2)$$

where  $\gamma_{ij}$  is the metric on a  $(n-2)$ -dimensional constant curvature space  $\mathcal{N}$  with its sectional curvature  $k = \pm 1, 0$ , and the two-dimensional spacetime  $\mathcal{T}$  spanned by the coordinates  $(v, r)$  possesses the metric as  $h_{ab}$ . In addition, during obtaining the generalized dynamical solutions, we first use the pure radiation as the matter field. Then we generalize the matter field to a more general case [30, 31], and hence obtain a generalized Vaidya-like solution, in which the generalized Vaidya solution appears as a special case. For the generalized Vaidya solution in dRGT massive gravity, we can find that it is consistent with the result in some previous work where the corresponding static solution has been found [18].

On the other hand, black hole thermodynamics (more generally gravi-thermodynamics) significantly boosts our understandings of gravity theory. It is even treated as a critical probe to the quantum gravity theory. Gravi-thermodynamics is well established in stationary spacetime. For the dynamical spacetime, there is still no generally accepted theory. The first difficulty is that some physical concepts, including temperature, entropy, horizon, etc, become subtle. The second difficulty is that it is hard to define a reversible process in a dynamical spacetime. However, some researches have shown that the unified first law is a nice approach in gravi-thermodynamics if the spacetime has an inner space with maximal symmetry, since usually it can be directly derived from the field equation itself [10, 31–35]. Thus, it can be applied in a dynamical spacetime without essential obstructions for these particular spacetime. In our paper, since the inner space also has the maximal symmetry, we can apply the unified first law to investigate the thermodynamics of the above generalized dynamical solutions in dRGT massive gravity. Note that, the Misner-Sharp mass is a significant quantity in the unified first law. In Einstein's general relativity, the Misner-Sharp mass always exist [10, 31–35]. In addition, since it encodes rich information of the corresponding gravity field [36], from which one can obtain a series of exact solutions through thermodynamic methods [37]. However, the generalized Misner-Sharp mass may be absent in some modified gravity like  $f(R)$  gravity [38, 39]. In our case, by using the unified first law, we can find that the generalized Misner-Sharp mass does exist for the above generalized dynamical solutions, and obtain the first law of thermodynamics on the apparent horizon for these generalized dynamical solutions. In addition, the usual Clausius relation  $\delta Q = TdS$  holds on the apparent horizon, which implies that the dRGT massive gravity is in a thermodynamic equilibrium state [10, 38, 40, 41]. It should be emphasized that the existence of the Misner-Sharp mass in some special solutions does not imply the existence of it in the corresponding gravity theory. For example, the Misner-Sharp mass exists in the FRW solution and static solution in  $f(R)$  gravity. However, it does not always exist in a general spherically symmetric spacetime in  $f(R)$  gravity [38, 39]. Essentially, the generalized Misner-Sharp mass is a conserved charge of the spacetime corresponding to the Kodama vector (reduced to a Killing one in stationary spacetime), which depends on the gravity theory in consideration [33, 38, 39]. The integrability of such a conserved charge, and thus the existence of the generalized Misner-Sharp mass, is a non-trivial problem. Therefore, we need further study the existence of the generalized Misner-Sharp mass in a general spacetime with maximally symmetric subspaces. We show that the generalized Misner-Sharp mass in the  $n$ -dimensional dRGT massive gravity indeed exist, and the covariant form has also been obtained, i.e., the result is not constrained to any special solution.

This paper is organized as follows. In section II, we first obtain the generalized Vaidya solution in the dRGT massive gravity, and then consider a more general matter field to obtain a generalized Vaidya-like solution. In section III, we use the unified first law to investigate the thermodynamics of these generalized dynamical solutions. Our results show that the generalized Misner-Sharp mass exist in these solutions. In Section IV, we further derive the covariant generalized Misner-Sharp mass for the  $n$ -dimensional dRGT massive gravity by considering the more general metric ansatz and matter field. Finally, we draw the conclusions and discussions in Section V.

## II. GENERALIZED DYNAMICAL SOLUTIONS IN THE $N$ -DIMENSIONAL MASSIVE GRAVITY

In this section, we explore the generalized dynamical solutions in the  $n$ -dimensional dRGT massive gravity. The action of the dRGT massive gravity in an  $n$ -dimensional spacetime with a cosmological constant  $\Lambda = -\frac{(n-1)(n-2)}{2\ell^2}$  reads [9, 18],

$$S = \frac{1}{16\pi G} \int d^n x \sqrt{-g} \left[ R + \frac{(n-1)(n-2)}{\ell^2} + m^2 \sum_i^4 c_i \mathcal{U}_i(g, f) \right], \quad (3)$$

where  $f$  is a constant symmetric tensor, which is usually called reference metric,  $c_i$  and  $\ell$  are constants, and  $\mathcal{U}_i$  are symmetric polynomials of the eigenvalues of the  $n \times n$  matrix  $\mathcal{K}^\mu{}_\nu \equiv \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$ ,

$$\begin{aligned} \mathcal{U}_1 &= [\mathcal{K}], \\ \mathcal{U}_2 &= [\mathcal{K}]^2 - [\mathcal{K}^2], \\ \mathcal{U}_3 &= [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3], \\ \mathcal{U}_4 &= [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]. \end{aligned} \quad (4)$$

The square root in  $\mathcal{K}$  means  $(\sqrt{A})^\mu{}_\nu (\sqrt{A})^\nu{}_\lambda = A^\mu{}_\lambda$  and  $[\mathcal{K}] = K^\mu{}_\mu = \sqrt{g^{\mu\alpha} f_{\alpha\mu}}$ .

From the action and considering the matter fields, the equations of motion are

$$\mathcal{G}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{(n-1)(n-2)}{2\ell^2} g_{\mu\nu} + m^2 \chi_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (5)$$

where

$$\begin{aligned} \chi_{\mu\nu} &= -\frac{c_1}{2} (\mathcal{U}_1 g_{\mu\nu} - \mathcal{K}_{\mu\nu}) - \frac{c_2}{2} (\mathcal{U}_2 g_{\mu\nu} - 2\mathcal{U}_1 \mathcal{K}_{\mu\nu} + 2\mathcal{K}_{\mu\nu}^2) - \frac{c_3}{2} (\mathcal{U}_3 g_{\mu\nu} - 3\mathcal{U}_2 \mathcal{K}_{\mu\nu} \\ &\quad + 6\mathcal{U}_1 \mathcal{K}_{\mu\nu}^2 - 6\mathcal{K}_{\mu\nu}^3) - \frac{c_4}{2} (\mathcal{U}_4 g_{\mu\nu} - 4\mathcal{U}_3 \mathcal{K}_{\mu\nu} + 12\mathcal{U}_2 \mathcal{K}_{\mu\nu}^2 - 24\mathcal{U}_1 \mathcal{K}_{\mu\nu}^3 + 24\mathcal{K}_{\mu\nu}^4). \end{aligned} \quad (6)$$

In this article, we will investigate the generalized dynamical solutions in the  $n$ -dimensional spacetime with a maximally symmetric inner space in the dRGT massive gravity, and the metric ansatz is just (2). For this metric ansatz, we take the following reference metric as in [18]

$$f_{\mu\nu} = \text{diag}(0, 0, c_0^2 \gamma_{ij}), \quad (7)$$

with  $c_0$  is a positive constant. Thus,

$$[\mathcal{K}] = \frac{n-2}{r} c_0, \quad [\mathcal{K}^2] = \frac{n-2}{r^2} c_0^2, \quad [\mathcal{K}^3] = \frac{n-2}{r^3} c_0^3, \quad [\mathcal{K}^4] = \frac{n-2}{r^4} c_0^4 \quad (8)$$

with the symmetric polynomials become

$$\mathcal{U}_1 = \frac{(n-2)c_0}{r}, \quad (9)$$

$$\mathcal{U}_2 = \frac{(n-2)(n-3)c_0^2}{r^2}, \quad (10)$$

$$\mathcal{U}_3 = \frac{(n-2)(n-3)(n-4)c_0^3}{r^3}, \quad (11)$$

$$\mathcal{U}_4 = \frac{(n-2)(n-3)(n-4)(n-5)c_0^4}{r^4}, \quad (12)$$

and the corresponding components of  $\mathcal{G}_{\mu\nu}$  are

$$\mathcal{G}_v^v = \mathcal{G}_r^r = \Lambda + \frac{n-2}{2} \times \left[ \frac{(r^{n-3}f)' - (n-3)r^{n-4}k - c_1c_0m^2r^{n-3} - (n-3)c_2c_0^2m^2r^{n-4} - (n-3)(n-4)c_3c_0^3m^2r^{n-5}}{r^{n-2}} - \frac{(n-3)(n-4)(n-5)c_4c_0^4m^2r^{n-6}}{r^{n-2}} \right], \quad (13)$$

$$\mathcal{G}_j^i = \delta_j^i \times \left[ \Lambda + \frac{(r^{n-3}f)'' - (n-3)(n-4)r^{n-5}k - (n-3)c_1c_0m^2r^{n-4} - (n-3)(n-4)c_2c_0^2m^2r^{n-5}}{2r^{n-3}} - \frac{(n-3)(n-4)(n-5)c_3c_0^3m^2r^{n-6} - (n-3)(n-4)(n-5)(n-6)c_4c_0^4m^2r^{n-7}}{2r^{n-3}} \right], \quad (14)$$

$$\mathcal{G}_v^r = \frac{-(n-2)\dot{f}}{2r}, \quad (15)$$

$$\mathcal{G}_r^v = 0. \quad (16)$$

where a prime/overdot denotes the derivative with respect to  $r/v$ . In the followings, we investigate two cases by considering different matter fields. In the first case, the generalized Vaidya solution is derived sourced by the pure radiations in analogy to the Vaidya solution in the Einstein gravity. In the second case, we consider a more generic source matter than the usual pure radiations to obtain a more general dynamical solution, i.e., a generalized Vaidya-like solution.

### A. Special case: Generalized Vaidya Solution

For the pure radiations, the stress energy tensor is given by  $T_{ab} = \mu l_a l_b$ , where  $l_a = (dv)_a$  is expressed in the coordinates  $(v, r, x^i)$  in (2). The components of the field equation (5) corresponding to the metric (2) present,

$$\mathcal{G}_v^v = \mathcal{G}_r^r = \Lambda + \frac{n-2}{2} \times \left[ \frac{(r^{n-3}f)' - (n-3)r^{n-4}k - c_1c_0m^2r^{n-3} - (n-3)c_2c_0^2m^2r^{n-4} - (n-3)(n-4)c_3c_0^3m^2r^{n-5}}{r^{n-2}} - \frac{(n-3)(n-4)(n-5)c_4c_0^4m^2r^{n-6}}{r^{n-2}} \right] = 0, \quad (17)$$

$$\mathcal{G}_j^i = \delta_j^i \times \left[ \Lambda + \frac{(r^{n-3}f)'' - (n-3)(n-4)r^{n-5}k - (n-3)c_1c_0m^2r^{n-4} - (n-3)(n-4)c_2c_0^2m^2r^{n-5}}{2r^{n-3}} - \frac{(n-3)(n-4)(n-5)c_3c_0^3m^2r^{n-6} - (n-3)(n-4)(n-5)(n-6)c_4c_0^4m^2r^{n-7}}{2r^{n-3}} \right] = 0, \quad (18)$$

$$\mathcal{G}_v^r = \frac{-(n-2)\dot{f}}{2r} = 8\pi G\mu, \quad (19)$$

$$\mathcal{G}_r^v = 0, \quad (20)$$

Note that the components  $\mathcal{G}_j^i$  are not independent, because they are linear combination of the terms of  $\mathcal{G}_v^v$  and  $\partial_r \mathcal{G}_v^v$ ,

$$\mathcal{G}_j^i = \delta_j^i [\mathcal{G}_v^v + r\partial_r \mathcal{G}_v^v / (n-2)] = \delta_j^i \left[ \frac{1}{(n-2)r^{n-3}} \partial_r (r^{n-2} \mathcal{G}_v^v) \right]. \quad (21)$$

Therefore,  $\mathcal{G}_j^i = 0$  do not yield independent equations. From the above equation in (17), we can easily obtain the generalized Vaidya solution in the  $n$ -dimensional dRGT massive gravity,

$$f(v, r) = k + \frac{r^2}{\ell^2} - \frac{M(v)}{r^{n-3}} + \frac{c_0c_1m^2}{n-2}r + c_0^2c_2m^2 + \frac{(n-3)c_0^3c_3m^2}{r} + \frac{(n-3)(n-4)c_0^4c_4m^2}{r^2}, \quad (22)$$

with

$$\mu = -\frac{(n-2)\dot{f}}{16\pi Gr} = \frac{(n-2)\dot{M}(v)}{16\pi Gr^{n-2}}, \quad (23)$$

which can be obtained by inserting (22) into (19), and  $M(v)$  is the mass parameter. Our solution is consistent with the result in some previous work like [18]. Since if  $M(v)$  is independent of  $v$ , i.e. a constant, and hence  $f(v, r)$  can be written as  $f(r)$ , then after the transformation in the metric ansatz (2)

$$dv = dt + \frac{1}{f(r)}dr, \quad (24)$$

the above solution (22) comes back to the static solution in  $n$ -dimensional spacetime found in [18].

### B. The general case: Generalized Vaidya-like Solution

Now we further generalize the above generalized Vaidya solution in the dRGT massive gravity to a more general case. Note that, for the metric (2) and the reference metric (7), we have  $\mathcal{G}_r^r = \mathcal{G}_v^v$ , so the energy-momentum tensor of matter field should satisfy  $T_r^r = T_v^v$ . Certainly, the pure radiation matter discussed in the above satisfies the constraint. In fact, they are  $T_r^r = T_v^v = 0$ . Therefore, if we relax this condition to  $T_i^i = \sigma T_r^r = \sigma T_v^v$  (where  $\sigma$  is a constant, and the equation does not sum over  $i$ ), then from the equation  $\nabla_\mu T_\nu^\mu = 0$  or the explicit expressions of  $\mathcal{G}_\nu^\mu$  in equations (13) to (15), we can derive

$$\partial_v T_v^v + \partial_r T_v^r + \frac{n-2}{r} T_v^r = 0, \quad (25)$$

and

$$\partial_r T_v^v + \frac{(n-2)(1-\sigma)}{r} T_v^v = 0. \quad (26)$$

So, for the pure radiation matter with  $T_r^r = T_v^v = 0$ , one finds that  $T_v^r$  has to be proportional to  $1/r^{n-2}$ , which is consistent with the above generalized Vaidya case in (23).

Therefore, for the more general case  $T_r^r = T_v^v \neq 0$  for the matter field, and hence from the equation (26), they should satisfy

$$T_r^r = T_v^v = \mathcal{C}(v) r^{-(n-2)(1-\sigma)}, \quad (27)$$

where  $\mathcal{C}(v)$  is a function of  $v$ . In addition, the off-diagonal part of the energy-momentum tensor  $T_\nu^\mu$ , i.e., the component  $T_v^r$  has to satisfy the equation (25). Now the equation (17) is modified as

$$\mathcal{G}_v^v = 8\pi G \mathcal{C}(v) r^{-(n-2)(1-\sigma)}. \quad (28)$$

Integrating this equation, we obtain the expression of  $f(v, r)$

$$\begin{aligned} f(v, r) = & k + \frac{r^2}{\ell^2} + \frac{c_0 c_1 m^2}{n-2} r + c_0^2 c_2 m^2 + \frac{(n-3)c_0^3 c_3 m^2}{r} + \frac{(n-3)(n-4)c_0^4 c_4 m^2}{r^2} \\ & - \frac{M(v)}{r^{n-3}} + \frac{16\pi G}{(n-2)r^{n-3}} \mathcal{C}(v) \Theta(r), \end{aligned} \quad (29)$$

where  $M(v)$  is an arbitrary function of  $v$ , and  $\Theta(r) = \int dr r^{(n-2)\sigma}$ . In details, when  $\sigma = -1/(n-2)$ ,

$$\Theta(r) = \ln(r), \quad (30)$$

and in other cases

$$\Theta(r) = \frac{r^{(n-2)\sigma+1}}{(n-2)\sigma+1}. \quad (31)$$

Note that, the parameter  $\sigma$ , functions  $m(v)$  and  $\mathcal{C}(v)$  should satisfy some consistency relations if one imposes some energy condition for the energy-momentum tensor. In addition, from (15), we have

$$T_v^r = \tilde{\mu} = \frac{(n-2)\dot{M}(v)}{16\pi G r^{n-2}} - \frac{\dot{\mathcal{C}}(v)\Theta(r)}{r^{n-2}}. \quad (32)$$

which is also consistent with the equation (25). Therefore, we have also obtained the stress tensor of matter field in this more general case. More precisely, we can further write the stress tensor of matter field in this more general case as

$$T_{ab} = \tilde{\mu} l_a l_b - P(l_a n_b + n_a l_b) + \sigma P q_{ab}, \quad (33)$$

where  $n_a$  is a null vector which satisfies  $l_a n^a = -1$ . In coordinates  $(v, r, x^i)$ ,  $l_a = (dv)_a$  and  $n_a = f/2(dv)_a - (dr)_a$ , while the tensor  $q_{ab}$  is a projection operator given by  $q_{ab} = g_{ab} + l_a n_b + l_b n_a$ , and the quantity  $P$  is the radial pressure with the form  $P = \mathcal{C}(v)r^{-(n-2)(1-\sigma)}$ . In addition, the metric (2) can be put into the form  $g_{ab} = h_{ab} + q_{ab}$ , where

$$h_{ab} = -l_a n_b - l_b n_a \quad (34)$$

is the metric of two-dimensional spacetime  $\mathcal{T}$  spanned by the coordinates  $(v, r)$ . Certainly, in the coordinates  $(v, r, x^i)$ , the line element of  $h_{ab}$  can be expressed as  $-f(v, r)dv^2 + 2dvdr$ . Therefore, (29) together with (33) is a more general case with new dynamical solution, which we call the generalized Vaidya-like solution. Obviously, the above generalized Vaidya solution is a special case of this generalized Vaidya-like solution with  $\mathcal{C}(v) = 0$ .

### III. THERMODYNAMICS OF THE GENERALIZED DYNAMICAL SOLUTIONS

In this section, we will investigate the thermodynamics of the above generalized dynamical solutions in the dRGT massive gravity by using the unified first law, and we concentrate on the generalized Vaidya-like solution obtained in the more general case, since it naturally includes the generalized Vaidya solution as a special case. According to the unified first law, similar to the case of Einstein gravity [32], one usually can formally cast the equation (5) of gravitational field into the form,

$$dM_{eff} = A\Psi_a dx^a + WdV, \quad (35)$$

where  $A = V_k r^{n-2}$  and  $V = V_k r^{n-1}/(n-1)$  are the area and volume of the  $(n-2)$ -dimensional constant curvature space  $\mathcal{N}$  with radius  $r$ ,  $W$  is called work density defined as  $W = -h^{ab}T_{ab}/2$  and  $\Psi_a$  is the energy supply vector with the definition  $\Psi_a = T_a^b \partial_b r + W \partial_a r$ . Here,  $T_{ab}$  is the projection of the stress tensor  $T_{\mu\nu}$  of matter into  $h_{ab}$ .

After substituting the explicit forms of generalized dynamical solutions in the dRGT massive gravity (29) and (33), we can explicitly obtain the following quantities,

$$W = -P, \quad \Psi_a = \tilde{\mu} l_a, \quad (36)$$

$$A\Psi_a dx^a + WdV = V_k r^{n-2} \tilde{\mu} dv - P V_k r^{n-2} dr \equiv X(v, r)dv + Y(v, r)dr. \quad (37)$$

It is easy to check,

$$\frac{\partial X(v, r)}{\partial r} = \frac{\partial Y(v, r)}{\partial v}, \quad (38)$$

which ensures that  $dM_{eff}$  is a closed form, and thus qualified as the generalized Misner-Sharp mass for the above generalized dynamical solutions in the dRGT massive gravity. Moreover, the generalized Misner-Sharp mass can be easily obtained in this case,

$$M_{eff} = V_k \left[ \frac{(n-2)M(v)}{16\pi G} - \mathcal{C}(v)\Theta(r) \right], \quad (39)$$

Next, we will use the unified first law and generalized Misner-Sharp mass (39) to investigate the thermodynamics of the above generalized dynamical solutions on the apparent horizon  $r_A$ , where  $r_A$  is defined as the trapped surface  $h^{ab}\partial_a r \partial_b r = 0$ . In our case, we can easily obtain the location of the apparent horizon  $r_A$  is  $f(v, r) = 0$  in (29). On the apparent horizon, the energy flow across the apparent horizon is [10, 31, 34, 35]

$$\delta Q = dM_{eff}|_{r_A} = A\Psi_a dx^a|_{r=r_A} = A\Psi_v dv = -\frac{(n-2)V_k r_A^{n-3}}{16\pi G} \dot{f}(r_A) dv. \quad (40)$$

On the other hand, the temperature of generalized dynamical solution is  $T = \frac{\kappa}{2\pi}$ , where the surface gravity  $\kappa$  defined on the apparent horizon is  $\kappa = D_a D^a r = \frac{1}{2\sqrt{-h}} \frac{\partial}{\partial x^\mu} (\sqrt{-h} h^{\mu\nu} \partial_\nu r) = f'(r_A)/2$  [10, 31–35]. Here,  $D_a$  is the covariant derivative associated with metric  $h_{ab}$ . In addition, the entropy of apparent horizon is  $S = \frac{A}{4G} = \frac{V_k r_A^{n-2}}{4G}$  [18]. Therefore,

$$TdS = \frac{\kappa}{2\pi} dS = \frac{(n-2)V_k r_A^{n-3}}{16\pi G} f'(r_A) \dot{r}_A dv. \quad (41)$$

After using the simple relation  $f'(r_A) \dot{r}_A = -\dot{f}(r_A)$  derived from  $f(r_A, v) = 0$ , we can easily obtain that the usual Clausius relation  $\delta Q = TdS$  does hold on the apparent horizon of the generalized dynamical solution, which indicates

that the dRGT massive gravity is an equilibrium state [41]. Note that, this result is consistent with the investigation in [10] by taking the FRW universe into account. In addition, it should be emphasized that the usual Clausius relation  $\delta Q = TdS$  does not always hold on the apparent horizon. For example, the usual Clausius relation does not hold for the  $f(R)$  gravity, which can be treated as the effects of the nonequilibrium of the space-time [38, 40, 41]. Therefore, after taking (41) and Clausius relation into account, the unified first law in (35) on the apparent can be rewritten as

$$dM_{\text{eff}} = TdS + WdV, \quad (42)$$

which is just the first law of thermodynamics for the generalized Vaidya-like solution. Note that, the work density  $W$  in (42) is nonzero for the generalized Vaidya-like solution, which makes another difference from the generalized Vaidya solution whose  $W = 0$ .

#### IV. GENERALIZED MISNER-SHARP MASS FOR THE $N$ -DIMENSIONAL MASSIVE GRAVITY

Note that, the Misner-Sharp mass is a quantity depending on not only the symmetry in the solution but also the underlying gravity theory, and hence the existence of Misner-Sharp mass in some special solution with maximally symmetric subspace does not always guarantee its existence in the gravity for the general solutions with the same maximally symmetric subspace, i.e.  $f(R)$  gravity [38, 39]. Therefore, we should further investigate the existence of the Misner-Sharp mass in a general spherically symmetric spacetime, although we have obtained the generalized Misner-Sharp mass for the above generalized dynamical solutions. In our case, in order to investigate the generalized Misner-Sharp mass for the  $n$ -dimensional dRGT massive gravity, we usually write down the more general metric ansatz in a double-null coordinates as follows,

$$ds^2 = -2e^{-\varphi(u,v)} du dv + r^2(u,v) \gamma_{ij} dx^i dx^j. \quad (43)$$

here  $\gamma_{ij}$  is the metric on the maximally symmetric subspace same as in (2). In the coordinates (43) the RHS in (35) reads,

$$A\Psi_a dx^a + WdV = A(u,v)du + B(u,v)dv, \quad (44)$$

where

$$A(u,v) = V_k r^{n-2} e^\varphi (r_{,u} T_{uv} - r_{,v} T_{uu}), \quad (45)$$

$$B(u,v) = V_k r^{n-2} e^\varphi (r_{,v} T_{uv} - r_{,u} T_{vv}). \quad (46)$$

Here a comma denotes partial derivative. Substituting (35) into (44), we reach

$$F \equiv dM_{\text{eff}} = A(u,v)du + B(u,v)dv. \quad (47)$$

The components of the field equation (5) in the coordinates (43) read,

$$\begin{aligned} 8\pi GT_{uu} &= -(n-2) \frac{\varphi_{,u} r_{,u} + r_{,uu}}{r}, \\ 8\pi GT_{vv} &= -(n-2) \frac{\varphi_{,v} r_{,v} + r_{,vv}}{r}, \\ 8\pi GT_{uv} &= \frac{-\Lambda}{e^\varphi} + \frac{n-2}{2e^\varphi r} (2r_{,uv} e^\varphi + c_1 c_0 m^2) + \frac{(n-2)(n-3)(k + 2e^\varphi r_{,u} r_{,v} + c_2 c_0^2 m^2)}{2e^\varphi r^2} \\ &\quad + \frac{(n-2)(n-3)(n-4)c_3 c_0^3 m^2}{2e^\varphi r^3} + \frac{(n-2)(n-3)(n-4)(n-5)c_4 c_0^4 m^2}{2e^\varphi r^4}. \end{aligned} \quad (48)$$

Obviously, a well-defined  $M_{\text{eff}}$  in (47) requires  $F$  is a closed form  $dF = 0$ , which means,

$$A_{,v} dv \wedge du + B_{,u} du \wedge dv = 0. \quad (49)$$

Then we obtain the constraint for a well-defined  $M_{\text{eff}}$ ,

$$A_{,v} = B_{,u}. \quad (50)$$



Substituting (48) into (45) and (46), we obtain

$$\begin{aligned}
A(u, v) &= \frac{V_k}{8\pi G} [-\Lambda r_{,u} r^{n-2} + (n-2)r^{n-3} e^\varphi r_{,u} r_{,vu} + \frac{k}{2}(n-2)(n-3)r^{n-4} r_{,u} + (n-2)(n-3)e^\varphi r_{,v} r_{,u}^2 r^{n-4} \\
&\quad + e^\varphi r^{n-3}(n-2)(r_{,u} r_{,v} + r_{,v} r_{,uv}) + \frac{(n-2)r^{n-3} r_{,u} c_1 c_0 m^2}{2} + \frac{(n-2)(n-3)r^{n-4} r_{,u} c_2 c_0^2 m^2}{2} \\
&\quad + \frac{(n-2)(n-3)(n-4)r^{n-5} r_{,u} c_3 c_0^3 m^2}{2} + \frac{(n-2)(n-3)(n-4)(n-5)r^{n-6} r_{,u} c_4 c_0^4 m^2}{2}], \\
B(u, v) &= \frac{V_k}{8\pi G} [-\Lambda r_{,v} r^{n-2} + (n-2)r^{n-3} e^\varphi r_{,v} r_{,vu} + \frac{k}{2}(n-2)(n-3)r^{n-4} r_{,v} + (n-2)(n-3)e^\varphi r_{,u} r_{,v}^2 r^{n-4} \\
&\quad + e^\varphi r^{n-3}(n-2)(r_{,v} r_{,u} + r_{,u} r_{,uv}) + \frac{(n-2)r^{n-3} r_{,v} c_1 c_0 m^2}{2} + \frac{(n-2)(n-3)r^{n-4} r_{,v} c_2 c_0^2 m^2}{2} \\
&\quad + \frac{(n-2)(n-3)(n-4)r^{n-5} r_{,v} c_3 c_0^3 m^2}{2} + \frac{(n-2)(n-3)(n-4)(n-5)r^{n-6} r_{,v} c_4 c_0^4 m^2}{2}]. \tag{51}
\end{aligned}$$

Using the above explicit forms of  $A(u, v)$  and  $B(u, v)$ , we find that the above constraint is automatically satisfied for the  $n$ -dimensional dRGT massive gravity, which guarantees that  $M_{eff}$  is well-defined. Thus directly integrating (35) presents the generalized Misner-Sharp mass in the  $n$ -dimensional dRGT massive gravity

$$\begin{aligned}
M_{eff} &= \int A(u, v) du + \int \left[ B(u, v) - \frac{\partial}{\partial v} \int A(u, v) du \right] dv \\
&= \frac{V_k(n-2)}{16\pi G} r^{n-3} \left[ \frac{r^2}{\ell^2} + k + 2e^\varphi r_{,u} r_{,v} + \frac{c_0 c_1 m^2}{n-2} r + c_0^2 c_2 m^2 + \frac{(n-3)c_0^3 c_3 m^2}{r} + \frac{(n-3)(n-4)c_0^4 c_4 m^2}{r^2} \right]. \tag{52}
\end{aligned}$$

Note that, here the second term in the first line of (52) in fact vanishes, and we have fixed an integration constant so that  $M_{eff}$  reduces to the Misner-Sharp mass in the Einstein gravity when the graviton mass parameter  $m$  goes to zero. Furthermore, the above generalized Misner-Sharp mass can be rewritten in a covariant form as

$$M_{eff} = \frac{V_k(n-2)}{16\pi G} r^{n-3} [(k - h^{ab} \partial_a r \partial_b r) + \frac{r^2}{\ell^2} + \frac{c_0 c_1 m^2}{n-2} r + c_0^2 c_2 m^2 + \frac{(n-3)c_0^3 c_3 m^2}{r} + \frac{(n-3)(n-4)c_0^4 c_4 m^2}{r^2}] \tag{53}$$

For the special case in the above generalized dynamical solution (29), one can check that the result in (39) is consistent with the generalized Misner-Sharp mass in (53). And (53) is the general definition of the generalized Misner-Sharp mass in the  $n$ -dimensional spacetime with maximally symmetric subspace in the dRGT massive gravity.

## V. CONCLUSION AND DISCUSSION

In this paper, through considering the pure radiation and a more general case as the matter fields, we obtain the generalized dynamical solutions in the  $n$ -dimensional dRGT massive gravity, which naturally includes the generalized Vaidya solution. By using the unified first law and Misner-Sharp mass, we investigate the thermodynamics for these solutions. Besides obtaining the first law of thermodynamics for these generalized dynamical solutions on the apparent horizon, we also check that the generalized Misner-Sharp mass exists for them. Generally, a solution has much higher symmetry than the theory itself. The existence of the Misner-Sharp mass in some special solutions does not imply the existence of it in the general theory. For example, the Misner-Sharp mass exists in the FRW solutions and static solutions in  $f(R)$  gravity. However, it does not always exist in a general spherically symmetric spacetime in  $f(R)$  gravity. In view of this situation, we further investigate the generalized Misner-Sharp by taking the general metric ansatz and matter field into account, and find that the generalized Misner-Sharp mass really exists in a covariant form.

Note that, in massive gravity theory, a reference metric is required. However, the theory itself does not determine the concrete form of the reference metric. This uncertainty makes the theory become arbitrary in some degree, while delivers extra conveniences in some cases. For example, there is no Schwarzschild solution in the unitary gauge (Minkowskian reference metric), and thus to match the tests in the solar system a chameleon mechanism is necessary. Recently, Li et al find that the Schwarzschild solution can be obtained if one give up the unitary gauge [42]. Other solutions have also been found by choosing different reference metric, i.e. the rotating black hole solution in the dRGT massive gravity [43]. Therefore, it is an interesting issue to find other solutions in the dRGT massive gravity by considering different reference metrics.

In addition, recently the dynamics of black holes and black branes have been found to be greatly simplified in the limit in which the number of spacetime dimensions  $N$  grows very large [44], therefore, more properties for the black



holes and black branes in the large  $N$  limit will also be an interesting issue to further investigate. On the other hand, according to the AdS/CFT correspondence, the Vaidya dynamical black branes can be related to the thermalization process of the strongly coupling field [45, 46], i.e. thermalization process of the quark-gluon plasma (QGP) produced in ultrarelativistic heavy-ion collisions at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC). Therefore, the underlying dual physics of our Vaidya-like dynamical black brane in (29) is also an interesting open question to be explored further.

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- [1] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A173 (1939) 211C232.
  - [2] K. Hinterbichler, Rev. Mod. Phys. **84**, 671 (2012) doi:10.1103/RevModPhys.84.671 [arXiv:1105.3735 [hep-th]].
  - [3] D. G. Boulware and S. Deser, Phys. Rev. D **6**, 3368 (1972).
  - [4] C. de Rham, Living Rev. Rel. **17**, 7 (2014) [arXiv:1401.4173 [hep-th]].
  - [5] C. de Rham and G. Gabadadze, Phys. Rev. D **82**, 044020 (2010) [arXiv:1007.0443 [hep-th]].
  - [6] C. de Rham, G. Gabadadze and A. J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011) [arXiv:1011.1232 [hep-th]].
  - [7] S. F. Hassan and R. A. Rosen, Phys. Rev. Lett. **108**, 041101 (2012) [arXiv:1106.3344 [hep-th]].
  - [8] S. F. Hassan, R. A. Rosen and A. Schmidt-May, JHEP **1202**, 026 (2012) [arXiv:1109.3230 [hep-th]].
  - [9] D. Vegh, arXiv:1301.0537 [hep-th].
  - [10] Y. P. Hu and H. Zhang, Phys. Rev. D **92**, no. 2, 024006 (2015) [arXiv:1502.00069 [hep-th]].
  - [11] H. Zhang and X. Z. Li, Phys. Rev. D **93**, no. 12, 124039 (2016) [arXiv:1510.03204 [gr-qc]].
  - [12] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].
  - [13] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].
  - [14] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].
  - [15] M. Blake, D. Tong and D. Vegh, Phys. Rev. Lett. **112**, no. 7, 071602 (2014) [arXiv:1310.3832 [hep-th]].
  - [16] A. Amoretti, A. Braggio, N. Maggiore, N. Magnoli and D. Musso, JHEP **1409**, 160 (2014) doi:10.1007/JHEP09(2014)160 [arXiv:1406.4134 [hep-th]].
  - [17] Y. P. Hu, H. F. Li, H. B. Zeng and H. Q. Zhang, Phys. Rev. D **93**, no. 10, 104009 (2016) [arXiv:1512.07035 [hep-th]].
  - [18] R. G. Cai, Y. P. Hu, Q. Y. Pan and Y. L. Zhang, Phys. Rev. D **91**, no. 2, 024032 (2015) [arXiv:1409.2369 [hep-th]].
  - [19] S. H. Hendi, S. Panahiyan and B. Eslam Panah, JHEP **1601**, 129 (2016) [arXiv:1507.06563 [hep-th]]; S. H. Hendi, B. Eslam Panah and S. Panahiyan, JHEP **1605**, 029 (2016) [arXiv:1604.00370 [hep-th]]; S. H. Hendi, N. Riazi and S. Panahiyan, arXiv:1610.01505 [hep-th].
  - [20] S. H. Hendi, B. E. Panah and S. Panahiyan, JHEP **1511**, 157 (2015) [arXiv:1508.01311 [hep-th]]; S. H. Hendi, B. E. Panah and S. Panahiyan, arXiv:1510.00108 [hep-th].
  - [21] Y. P. Hu, X. X. Zeng and H. Q. Zhang, arXiv:1611.00677 [hep-th].
  - [22] J. Xu, L. M. Cao and Y. P. Hu, Phys. Rev. D **91**, no. 12, 124033 (2015) [arXiv:1506.03578 [gr-qc]].
  - [23] L. M. Cao and Y. Peng, arXiv:1509.08738 [hep-th]; L. M. Cao, Y. Peng and Y. L. Zhang, arXiv:1511.04967 [hep-th].
  - [24] R. A. Davison, Phys. Rev. D **88**, 086003 (2013) [arXiv:1306.5792 [hep-th]].
  - [25] M. Blake and D. Tong, Phys. Rev. D **88**, no. 10, 106004 (2013) [arXiv:1308.4970 [hep-th]].
  - [26] R. A. Davison, K. Schalm and J. Zaanen, Phys. Rev. B **89**, 245116 (2014) [arXiv:1311.2451 [hep-th]].
  - [27] A. Adams, D. A. Roberts and O. Saremi, arXiv:1408.6560 [hep-th].
  - [28] T. Q. Do, Phys. Rev. D **93**, no. 10, 104003 (2016) [arXiv:1602.05672 [gr-qc]]; T. Q. Do, Phys. Rev. D **94**, no. 4, 044022 (2016) [arXiv:1604.07568 [gr-qc]].
  - [29] P. C. Vaidya, Proc. Indian Acad. Sci. **A33**, 264 (1951); R. W. Lindquist, R. A. Schwartz, and C. W. Misner, Phys. Rev. **137**, B1364 (1965).
  - [30] A. E. Dominguez and E. Gallo, Phys. Rev. D **73**, 064018 (2006) doi:10.1103/PhysRevD.73.064018 [gr-qc/0512150].
  - [31] R. G. Cai, L. M. Cao, Y. P. Hu and S. P. Kim, Phys. Rev. D **78**, 124012 (2008) [arXiv:0810.2610 [hep-th]].

- [32] S. A. Hayward, Phys. Rev. D **49**, 6467 (1994); S. A. Hayward, Phys. Rev. D **49**, 831 (1994) doi:10.1103/PhysRevD.49.831 [gr-qc/9303030]; S. A. Hayward, Phys. Rev. D **53**, 1938 (1996) [arXiv:gr-qc/9408002]; S. A. Hayward, Class. Quant. Grav. **15**, 3147 (1998) [arXiv:gr-qc/9710089].
- [33] H. Maeda and M. Nozawa, Phys. Rev. D **77**, 064031 (2008) [arXiv:0709.1199 [hep-th]].
- [34] R. G. Cai and S. P. Kim, JHEP **0502**, 050 (2005) [arXiv:hep-th/0501055].
- [35] R. G. Cai, L. M. Cao and Y. P. Hu, JHEP **0808**, 090 (2008) [arXiv:0807.1232 [hep-th]].
- [36] H. Zhang, The Universe **3** (2015) no.1, 30;
- [37] H. Zhang, S. Hayward, X. H. Zhai, and X.Z. Li, Phys. Rev. D **89**(2014)064052; H. Zhang and X. Z. Li, Phys. Lett. B **737** (2014) 395 [arXiv:1406.1553 [gr-qc]]; Hongsheng Zhang, Dao-Jun Liu, and Xin-Zhou Li, Phys. Rev. D **90**, 124051 (2014)[arxiv: 1405.7530]; D. He and Q. y. Cai, arXiv:1609.05825 [hep-th]; H. W. Tan, J. B. Yang, T. M. He and J. Y. Zhang, arXiv:1609.04181 [gr-qc].
- [38] R. G. Cai, L. M. Cao, Y. P. Hu and N. Ohta, Phys. Rev. D **80**, 104016 (2009) [arXiv:0910.2387 [hep-th]];
- [39] H. Zhang, Y. Hu and X. Z. Li, Phys. Rev. D **90**, no. 2, 024062 (2014) [arXiv:1406.0577 [gr-qc]].
- [40] M. Akbar and R. G. Cai, Phys. Rev. D **75**, 084003 (2007) [arXiv:hep-th/0609128]; M. Akbar and R. G. Cai, Phys. Lett. B **635**, 7 (2006) [arXiv:hep-th/0602156]; M. Akbar and R. G. Cai, Phys. Lett. B **648**, 243 (2007) [arXiv:gr-qc/0612089].
- [41] C. Eling, R. Guedens and T. Jacobson, Phys. Rev. Lett. **96**, 121301 (2006) [gr-qc/0602001]; T. Jacobson, Phys. Rev. Lett. **75**, 1260 (1995) [gr-qc/9504004].
- [42] P. Li, X. z. Li and P. Xi, Phys. Rev. D **93** (2016) no.6, 064040 doi:10.1103/PhysRevD.93.064040 [arXiv:1603.06039 [gr-qc]]; P. Li, X. z. Li and P. Xi, Class. Quant. Grav. **33**, no. 11, 115004 (2016) doi:10.1088/0264-9381/33/11/115004 [arXiv:1503.08952 [gr-qc]].
- [43] E. Babichev and A. Fabbri, Phys. Rev. D **90**, 084019 (2014) doi:10.1103/PhysRevD.90.084019 [arXiv:1406.6096 [gr-qc]].
- [44] R. Emparan, R. Suzuki and K. Tanabe, Phys. Rev. Lett. **115**, no. 9, 091102 (2015) doi:10.1103/PhysRevLett.115.091102 [arXiv:1506.06772 [hep-th]]; R. Emparan, K. Izumi, R. Luna, R. Suzuki and K. Tanabe, JHEP **1606**, 117 (2016) doi:10.1007/JHEP06(2016)117 [arXiv:1602.05752 [hep-th]].
- [45] V. Balasubramanian *et al.*, Phys. Rev. Lett. **106**, 191601 (2011).
- [46] V. Balasubramanian *et al.*, Phys. Rev. D **84**, 026010 (2011); V. Balasubramanian and S. F. Ross, Phys. Rev. D **61**, 044007 (2000).